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SOME REMARKS ON THE WEIGHTS OF UNKNOWNS AS DETERMINED BY THE METHOD OF DIFFERENTIAL CORRECTIONS

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In many problems of practical astronomy we apply the method of least square to non-linear equations. The computation is usually done by the method of differential corrections. If the equations of condition are

$$f_i(x, y, \ldots) = L_i, \qquad i/1 \ldots n, \tag{1}$$

where x, y ... are the unknown parameters to be determined and L_i are the observed quantities, the procedure is the following. We start with certain approximate values for x, y—let us denote them by x_0, y_0. For simplicity we limit our discussion to the case of two unknown parameters. We determine the corrections Δx and Δy by the least squares method from the equations

$$\frac{\partial f_i}{\partial x}\Delta x + \frac{\partial f_i}{\partial y}\Delta y = L_i - f_i, \tag{2}$$

with x and y in f_i, $\partial f_i/\partial x$, $\partial f_i/\partial y$ equal to x_0, y_0. We add Δx and Δy to x_0 and y_0 and, in theory at least, we repeat this procedure until the resulting corrections become equal zero. We shall assume in what follows that this can be achieved. In other words, we make the assumption that the whole procedure is convergent. It is easily seen that the values of x and y which we obtain in this way satisfy the equations

$$\left.\begin{aligned} \sum_{i/1}^{n} (L_i - f_i)\frac{\partial f_i}{\partial x} &= 0 \\ \sum_{i/1}^{n} (L_i - f_i)\frac{\partial f_i}{\partial y} &= 0. \end{aligned}\right\} \tag{3}$$

Equations (3) are, in fact, nothing but the necessary conditions making the sum of the squares of the residuals a minimum.

When solving (2) by the method of least squares we get the mean errors of the corrections Δx and Δy. These errors are characteristic for Δx and Δy only in so far as we regard Δx and Δy as defined by the normal equations pertaining to equations (2). In almost all cases known to me it is assumed (at least tacitly) that these errors represent also the mean errors of the solution obtained for x and y. It can be easily shown that, in general, this assumption is not valid.*

The inverse weights of Δx and Δy are equal to

$$\sum_{j/1}^{n}\left(\frac{\partial\,\Delta x}{\partial L_j}\right)^2 \quad\text{and}\quad \sum_{j/1}^{n}\left(\frac{\partial\,\Delta y}{\partial L_j}\right)^2,$$

respectively. We determine $\partial\,\Delta x/\partial L_j$ and $\partial\,\Delta y/\partial L_j$ by differentiating with respect to L_j the normal equations pertaining to equations (2). We get (the square brackets are used to denote the summation over i from 1 to n)

$$\left.\begin{aligned}\left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial x}\right]\frac{\partial\,\Delta x}{\partial L_j}+\left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right]\frac{\partial\,\Delta y}{\partial L_j}&=\frac{\partial f_j}{\partial x},\\ \left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right]\frac{\partial\,\Delta x}{\partial L_j}+\left[\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right]\frac{\partial\,\Delta y}{\partial L_j}&=\frac{\partial f_j}{\partial y}.\end{aligned}\right\}\tag{4}$$

By a well-known theorem[1] we find the inverse weights as the diagonal elements of the inverse of the matrix of the coefficients of equations (4).

In order to find the quantities $\partial x/\partial L_j$ and $\partial y/\partial L_j$ that are prerequisite for the determination of the weights of x and y we differentiate equations (3) with respect to L_j and we get

$$\left.\begin{aligned}&\left\{\left[\frac{\partial^2 f}{\partial x^2}(f-L)\right]+\left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial x}\right]\right\}\frac{\partial x}{\partial L_j}+\left\{\left[\frac{\partial^2 f}{\partial x\partial y}(f-L)\right]+\right.\\ &\qquad\qquad\left.\left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right]\right\}\frac{\partial y}{\partial L_i}=\frac{\partial f_j}{\partial x}\\ &\left\{\left[\frac{\partial^2 f}{\partial x\partial y}(f-L)\right]+\left[\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right]\right\}\frac{\partial x}{\partial L_j}+\left\{\left[\frac{\partial^2 f}{\partial y^2}(f-L)\right]+\right.\\ &\qquad\qquad\left.\left[\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\right]\right\}\frac{\partial y}{\partial L_j}=\frac{\partial f_i}{\partial y}.\end{aligned}\right\}\tag{5}$$

Now, by comparing (4) and (5) we see that the coefficients of the unknowns are not identical in both systems. They are identical when $[(\partial^2 f/\partial x^2)(f-L)]$, $[(\partial^2 f/\partial x\partial y)(f-L)]$, $[(\partial^2 f/\partial y^2)(f-L)]$ equal zero. But these conditions are not fulfilled in general. Even when the residuals $f_i - L_i$ are small and do not show any systematic behavior the considered terms may be, in cases, quite appreciable. If we have used equations (4) for the determination of the mean errors of Δx and Δy, we should check whether the sums $[(\partial^2 f/\partial x^2)(f - L)]$, $[(\partial^2 f/\partial x \partial y)(f - L)]$, ... are really negligible before we ascribe these errors to the unknowns themselves.† That this precaution is essential, we can see from the following numerical example.

Let us suppose that for 5 equidistant values of t: 0, 1, 2, 3, 4, we have observed the values of L, which is supposed to be given by the expression

$$tx + (t - 1)^2 y - (t - 1)(t - 3)xy = L. \qquad (6)$$

Let the observed values be: -8.0, $+4.0$, $+9.5$, $+13.5$, $+11.0$. After some trials, we find by the method of differential correction that the solution is: $x = 3$, $y = 1$. The residuals are: 0.0, $+1.0$, -0.5, $+0.5$, -1.0. The matrix of the coefficients of the system (4) is

$$\begin{pmatrix} 29, & 48 \\ 48, & 96 \end{pmatrix}.$$

Its inverse is

$$\begin{pmatrix} 0.2, & -0.1 \\ -0.1, & 0.0604 \end{pmatrix}.$$

Hence, denoting by $p_{\Delta x}$ the weight of Δx, by $p_{\Delta y}$ the weight of Δy (with $x = 3$, $y = 1$, both Δx and Δy are equal 0) we obtain

$$p_{\Delta x}^{-1} = 0.2; \qquad p_{\Delta y}^{-1} = 0.0604. \qquad (7)$$

If we would ascribe these weights to x and y themselves (as is generally done), we would run into a contradiction. We can write (6) in the form

$$A + Bt + Ct^2 = L, \qquad (8)$$

with

$$A = y(1 - 3x), \qquad B = x - 2y + 4xy, \qquad C = y(1 - x). \quad (9)$$

Eliminating x and y between the three equations (9), we get

$$A - C - AB + 3CB + 2AC - A^2 + 3C^2 = 0. \qquad (10)$$

Now we can treat A, B, C, as auxiliary (unknown) parameters of the problem and determine them by the least squares method from equations (8) linear in A, B, C, ($t = 0, 1, \ldots$; $L = -8.0, +4.0, \ldots$) with the condition (10) to be satisfied exactly. It may be done by a standard method[2] and we get (returning from A, B, C, to x and y), $x = 3$, $y = 1$, the same values obtained by the method of differential corrections. The weights of x and y, p_x, p_y are, however

$$p_x^{-1} = 0.0916; \qquad p_y^{-1} = 0.0277, \qquad (11)$$

more than two times greater than those given by (7). The discrepancy is of course spurious. If we determine $\partial x/\partial L_j$, $\partial y/\partial L_j$ from equations (5) and form the sums of their squares we obtain exactly the values given by (11).

Summary.—If one uses the method of differential corrections in a least squares solution, the mean errors of the differential corrections to the unknowns are equal to the mean errors of the unknowns themselves only in the special case when the sums of the products of the residuals by the second order partial derivatives of the functions figuring in the equations of the problem are negligible. This is so regardless of how small the differential corrections happen to be. If the sums are not negligible, the equations of the form (5) should be used when determining the weights of the unknowns.

* An extensive discussion of this and related problems is to be found in a paper by E. B. Wilson and R. R. Puffer, "Least Squares and Laws of Population Growth," *Proc. Amer. Acad. Arts Sci.*, **68,** No. 9 (1933). Cf., in particular, equations (25) and (26) and the considerations in the Appendix.

† Evidently the functions of the type $f(x, y, t)$ which fulfill the system of equations

$$\begin{aligned}\partial^2 f/\partial x^2 &= a(x, y)\,(\partial f/\partial x) + b(x, y)\,(\partial f/\partial y),\\ \partial^2 f/\partial x\partial y &= a'(x, y)\,(\partial f/\partial x) + b'(x, y)\,(\partial f/\partial y),\\ \partial^2 f/\partial y^2 &= a''(x, y)\,(\partial f/\partial x) + b''(x, y)\,(\partial f/\partial y),\end{aligned}$$

where $a, b;\ a', b';\ a'', b''$ are arbitrary functions of x and y, independent of t, will have the property of yielding in the standard least squares solution for differential corrections the correct values of the mean errors (t is used instead of subscript i).

[1] Whittaker, E. T., and Robinson, G., *The Calculus of Observations*, London, 1932, p. 241.

[2] *Ibid.*, p. 252.